Integer part of one sum.

https://www.linkedin.com/feed/update/urn:li:activity:6636686163107282944

The numerical sequence $x_1, x_2, \ldots,$ satisfies $x_1 = \dfrac{1}{2}$ and $x_{k+1} = x_k^2 + x_k$ for all $k \in \mathbb{N}$. Find the integer part of the sum

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \ldots + \frac{1}{x_{100}+1}.$$

(A. Andjans, Riga, 7 points)

Solution by Arkady Alt, San Jose, California, USA.

Since $x_{k+1} = x_k^2 + x_k \iff \dfrac{1}{x_{k+1}} = \dfrac{1}{x_k} - \dfrac{1}{x_k+1} \iff \dfrac{1}{x_k+1} = \dfrac{1}{x_k} - \dfrac{1}{x_{k+1}}$

then $S_n := \displaystyle\sum_{k=1}^{n} \frac{1}{x_k+1} = \sum_{k=1}^{n}\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) = \frac{1}{x_1} - \frac{1}{x_{n+1}} = 2 - \frac{1}{x_{n+1}}$.

Noting that $x_{k+1} - x_k = x_k^2 > 0$ we can conclude that x_k increase and, therefore, $x_{k+1} - x_k \geq x_1^2 = \dfrac{1}{4}, k \in \mathbb{N}$ implies $x_k \geq x_1 + \dfrac{k-1}{4} = \dfrac{k+1}{4}, k \in \mathbb{N}$.

Hence, $2 - \dfrac{1}{x_{n+1}} \geq 2 - \dfrac{4}{n+2}$ and, therefore, $2 - \dfrac{4}{n+2} \leq S_n < 2,\ \forall k \in \mathbb{N}$.

Since $1 \leq S_n < 2$ for any $n \geq 2$ then $\lfloor S_n \rfloor = 1$ for any $n \geq 2$ and in particular

$$\left\lfloor \frac{1}{x_1+1} + \frac{1}{x_2+1} + \ldots + \frac{1}{x_{100}+1} \right\rfloor = 1.$$